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ON WEAK CONVERGENCE OF EMPIRICAL PROCESSES FOR RANDOM NUMBER OF INDEPENDENT STOCHASTIC VECTORS

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CHAPEL HILL, NORTH CAROLINA

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FOREWORD

This is an interim report of the work done under Contract F 33615-71-C-1927 by the University of North Carolina. The work done by Pranab Kumar Sen in this report is sponsored by the Aerospace Research Laboratories under the above contract; it was accomplished on Project 7071, "Research in Applied Mathematics" and is technically monitored by P. R. Krishnaiah of the Aerospace Research Laboratories.

ABSTRACT

By the use of a semi-martingale property of the Kolmogorov supremum, the results of Pyke [Proc. Cambridge Phil. Soc. 64 (1968), 155-160] on the weak convergence of the empirical process with random sample size are simplified and extended to the case of $p(>1)$-dimensional stochastic vectors.

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1. Introduction. Consider a sequence $\{\underset{\sim}{X}_i=(X_{i1},\ldots,X_{ip})', i\geq 1\}$ of independent and identically distributed stochastic $p(\geq 1)$-vectors, defined on a probability space $(\Omega,\mathcal{A},P)$, with each $\underset{\sim}{X}_i$ having a continuous distribution function (df) $F(\underset{\sim}{x})$, $\underset{\sim}{x}\varepsilon R^p$, the p-dimensional Euclidean space. We denote the marginal df of X_{ij} by $F_{[j]}$, let $Y_{ij} = F_{[j]}(X_{ij})$, $j=1,\ldots,p$, $\underset{\sim}{Y}_i = (Y_{i1},\ldots,Y_{ip})'$, $i\geq 1$, $\underset{\sim}{t}=(t_1,\ldots,t_p)'$, and define

$$(1.1)\qquad G(\underset{\sim}{t}) = P\{Y_{ij}\leq t_j,\ j=1,\ldots,p\},\ \underset{\sim}{t}\varepsilon E^p,$$

where $E^p = \{\underset{\sim}{t}:\ 0\leq t_j\leq 1,\ i=1,\ldots,p\}$. Then, the empirical df for $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ is defined by

$$(1.2)\qquad G_n(\underset{\sim}{t}) = n^{-1}\sum_{i=1}^{n} c(\underset{\sim}{t}-\underset{\sim}{Y}_i),\ \underset{\sim}{t}\varepsilon E^p,$$

where $c(\underset{\sim}{u})=1$ iff $u_j\geq 0$, $j=1,\ldots,p$; otherwise, $c(\underset{\sim}{u})=0$. Consider then the empirical process

$$(1.3)\qquad W_n(\underset{\sim}{t}) = n^{\frac{1}{2}}[G_n(\underset{\sim}{t})-G(\underset{\sim}{t})],\ \underset{\sim}{t}\varepsilon E^p,$$

and denote by

$$(1.4)\qquad W_n = \{W_n(\underset{\sim}{t}):\ \underset{\sim}{t}\varepsilon E^p\}.$$

For $p=1$, it is well-known that W_n weakly converges to a Brownian motion $W^o=\{W^o(t):\ 0\leq t\leq 1\}$. For $p\geq 1$, on the space $D^p[0,1]$ of all real functions on E^p with no discontinuities of the second kind, W_n converges in distribution (in the (extended) Skorokhod J_1-topology) to an appropriate Gaussian function,

say, $W = \{W(\underset{\sim}{t}): \underset{\sim}{t} \in E^p\}$, where $E[W(\underset{\sim}{t})]=0$, and

$$(1.5) \qquad E[W(\underset{\sim}{s})W(\underset{\sim}{t})] = G(\underset{\sim}{t} \wedge \underset{\sim}{s}) - G(\underset{\sim}{t})G(\underset{\sim}{s}),\ \underset{\sim}{t},\ \underset{\sim}{s} \in E^p,$$

and $\underset{\sim}{t} \wedge \underset{\sim}{s} = (t_1 \wedge s_1, \ldots, t_p \wedge s_p)$, where $a \wedge b = \min(a,b)$; we refer to Neuhaus (1971) who also reviews the earlier literature.

Let now $\{N_\nu,\ \nu \geq 1\}$ be a sequence of positive integer-valued random variables, such that

$$(1.6) \qquad \nu^{-1} N_\nu \to \xi, \text{ in probability, as } \nu \to \infty,$$

where ξ is a positive random variable defined on the same probability space $(\Omega, \mathcal{A}, P)$.

For p=1, Pyke (1968) has shown that under (1.6), W_{N_ν} converges in law to W^o; his result is extended here to the general multivariate case.

Theorem 1. Under (1.6), for every $p \geq 1$,

$$W_{N_\nu} \xrightarrow{\mathcal{L}} W, \text{ in the Skorokhod } J_1\text{-typology on } D^p[0,1].$$

The proof is outlined in section 3. Whereas, Pyke's arguments rely heavily on the properties of an equivalently defined Poisson process (which may become quite complicated for p>1), our approach is based on a simple semi-martingale property of the Kolmogorov supremum, which is considered first in section 2.

2. Some preliminary results. For two real valued functions $Z(\underset{\sim}{t})$ and $Z^*(\underset{\sim}{t})$, defined on E^p, we let

$$(2.1) \qquad \rho(Z,Z^*) = \operatorname{Sup}\{|Z(\underset{\sim}{t}) - Z^*(\underset{\sim}{t})|: \underset{\sim}{t} \in E^p\},$$

and for every $n \geq 1$, let

(2.2) $$W_n^+ = \text{Sup}\{W_n(\underset{\sim}{t}):\ \underset{\sim}{t}\varepsilon E^P\},\ W_n^- = \text{Sup}\{-W_n(\underset{\sim}{t}):\ \underset{\sim}{t}\varepsilon E^P\};$$

(2.3) $$W_n^* = \max\{W_n^+, W_n^-\} = \rho(W_n, 0).$$

Let $\mathcal{F}_n$ be the σ-field generated by $\{X_1,\ldots,X_n\}$, so that $\mathcal{F}_n$ is ↑ in n (≥ 1). Then, we have the following

<u>Lemma 2.1. $\{n^{\frac{1}{2}}W_n^+,\ \mathcal{F}_n;\ n\geq 1\}$ and $\{n^{\frac{1}{2}}W_n^-,\ \mathcal{F}_n;\ n\geq 1\}$ are both non-negative semi-martingale sequences.</u>

<u>Proof</u>. We only prove the result for W_n^+, as the other follows similarly. Note that W_n^+ is, by definition, non-negative [as $W_n(\underset{\sim}{t})=0$ for $\underset{\sim}{t}=\underset{\sim}{0}$ or $\underset{\sim}{t}=\underset{\sim}{1}$]. Let $\underset{\sim}{t}_n^o(\varepsilon E^P)$ be a point such that

(2.4) $$W_n^+ = W_n(\underset{\sim}{t}_n^o);\ \underset{\sim}{t}_n^o \text{ need not be unique.}$$

Then, by definition,

(2.5) $$(n+1)^{\frac{1}{2}}W_{n+1}^+ = (n+1)^{\frac{1}{2}} \sup_{\underset{\sim}{t}\varepsilon E^P} W_{n+1}(\underset{\sim}{t}) \geq (n+1)^{\frac{1}{2}}W_{n+1}(\underset{\sim}{t}_n^o),$$

so that, by (1.2), (1.3) and (2.5), for every $n\geq 1$,

(2.6) $$\begin{aligned} E\{(n+1)^{\frac{1}{2}}W_{n+1}^+ \mid \mathcal{F}_n\} &\geq E\{(n+1)^{\frac{1}{2}}W_{n+1}(\underset{\sim}{t}_n^o) \mid \mathcal{F}_n\} \\ &= \textstyle\sum_{i=1}^{n+1} E\{[c(\underset{\sim}{t}_n^o - \underset{\sim}{Y}_i) - G(\underset{\sim}{t}_n^o)] \mid \mathcal{F}_n\} \\ &= \textstyle\sum_{i=1}^{n} [c(\underset{\sim}{t}_n^o - \underset{\sim}{Y}_i) - G(\underset{\sim}{t}_n^o)] + E\{[c(\underset{\sim}{t}_n^o - \underset{\sim}{Y}_{n+1}) - G(\underset{\sim}{t}_n^o)] \mid \mathcal{F}_n\} \\ &= n^{\frac{1}{2}}W_n^+ + 0 = n^{\frac{1}{2}}W_n^+, \end{aligned}$$

as, given $\mathcal{F}_n$, $c(\underset{\sim}{t}_n^0 - \underset{\sim}{Y}_{n+1})$ assumes the values 1 and 0 with respective conditional probabilities $G(\underset{\sim}{t}_n^0)$ and $1-G(\underset{\sim}{t}_n^0)$. Q.E.D.

Lemma 2.2. *For every* $n \geq 1$, *there exist two positive constants* c_0 *and* c_1, *independent of* n, *such that*

$$(2.7) \qquad E\{(W_n^+)^2\} \leq c_0/c_1 \text{ and } E\{(W_n^-)^2\} \leq c_0/c_1.$$

Proof. By partial integration,

$$(2.8) \qquad E\{(W_n^+)^2\} = 2\int_0^\infty xP\{W_n^+ > x\}dx,$$

where by Theorem 1 of Kiefer and Wolfowitz (1958), for all $n \geq 1$,

$$(2.9) \qquad P\{W_n^+ > x\} < c_0 \exp\{-c_1 x^2\} \text{ for all } x \geq 0.$$

Consequently, by (2.8) and (2.9), $E\{(W_n^+)^2\} \leq c_0/c_1$. The other result follows similarly.

Lemma 2.3. *For every* $\varepsilon > 0$, *there exists a positive* $K_\varepsilon(<\infty)$, *such that for every* $n \geq 1$,

$$(2.10) \qquad P\{\max_{1 \leq k \leq n} (k/n)^{\frac{1}{2}} \rho(W_k, 0) > K_\varepsilon\} < \varepsilon.$$

Proof. By (2.3), for every $\varepsilon > 0$,

$$(2.11) \qquad P\{\max_{1 \leq k \leq n} (k/n)^{\frac{1}{2}} \rho(W_k, 0) > K_\varepsilon\}$$

$$\leq P\{\max_{1 \leq k \leq n} k^{\frac{1}{2}} W_k^+ > n^{\frac{1}{2}} K_\varepsilon\} + P\{\max_{1 \leq k \leq n} k^{\frac{1}{2}} W_k^- > n^{\frac{1}{2}} K_\varepsilon\},$$

and hence, by Lemma 2.1 along with the Kolmogorov inequality for semi-martingales

[viz., Feller (1966, p. 235)], the right and side of (2.11) is bounded above by

$$(2.12) \qquad (nK_\varepsilon^2)^{-1}[nE\{(W_n^+)^2\} + nE\{(W_n^-)^2\}]$$

$$= [E\{(W_n^+)^2\} + E\{(W_n^-)^2\}]/K_\varepsilon^2$$

$$\leq 2c_0/c_1K_\varepsilon^2, \text{ by Lemma 2.2.}$$

The proof then follows by selecting $K_\varepsilon > [2c_0/c_1\varepsilon]^{\frac{1}{2}}$. Q.E.D.

Lemma 2.4. (Uniform continuity in probability). For every $\varepsilon>0$ and $\eta>0$, there exists a $\delta>0$ and an $n_0(\varepsilon,\eta)$, such that for $n \geq n_0(\varepsilon,\eta)$,

$$(2.13) \qquad P\{\max_{k:\ |k-n|<\delta n} \rho(W_k, W_n) > \varepsilon\} < \eta.$$

Proof. Proceeding as in the proof of Theorem 2.1 of Pyke (1968), namely, as in his (2.7) through (2.10), we are only to show that as $n \to \infty$,

$$(2.14) \qquad \max_{1 \leq k \leq n} (k/n)^{\frac{1}{2}} \rho(W_k, 0) = O_p(1),$$

$$(2.15) \qquad \rho(W_n, 0) = \operatorname{Sup}\{|W_n(\underset{\sim}{t})| : \underset{\sim}{t} \in E^p\} = O_p(1).$$

Now, (2.14) has already been proved in Lemma 2.3, while by Theorem 3.1 of Neuhaus (1971) along with his treatment on the weak convergence of W_n to W, it follows that for every $\varepsilon>0$, there exists a positive $M_\varepsilon(<\infty)$, such that

$$(2.16) \qquad \lim_{n\to\infty} P\{\rho(W_n, 0) > M_\varepsilon\}$$

$$= P\{\rho(W, 0) > M_\varepsilon\} < \varepsilon'; \quad 0<\varepsilon'<\varepsilon,$$

which completes the proof of the lemma.

We now show that $\{W_n\}$ is a mixing sequence in the sense of Rényi (1958). This follows by defining

(2.17) $$W_n'(\underset{\sim}{t}) = n^{-1}\{\sum_{i=k_n}^{n}[c(\underset{\sim}{t}-Y_i)-G(\underset{\sim}{t})]\}.\ \underset{\sim}{t}\in E^p,$$

where $k_n\to\infty$ but $n^{-\frac{1}{2}}k_n \to 0$ as $n\to\infty$, and noting that

(2.18) $$\rho(W_n,W_n') \leq n^{-\frac{1}{2}}k_n \to 0 \quad \text{as} \quad n\to\infty.$$

Consequently, proceeding as in the proof of Lemma 3 of Blum, Hanson and Rosenblatt (1963), we obtain from Lemma 2.4, the following.

<u>Lemma 2.5.</u> <u>If</u> $A\in\mathcal{A}$, <u>then for every</u> $\varepsilon>0$ <u>and</u> $\eta>0$, <u>there exists a</u> $\delta>0$, <u>such that</u> <u>as</u> $n\to\infty$,

(2.19) $$P\{\max_{k:\ |k-n|<\delta n} \rho(W_k,W_n) > \varepsilon|A\} < \eta.$$

Let us now define

(2.20) $$\omega_\delta(W_n) = \mathrm{Sup}\{|W_n(\underset{\sim}{t})-W_n(\underset{\sim}{t}')|:\ |\underset{\sim}{t}-\underset{\sim}{t}'|<\delta\}.$$

Then, from the results of section 5 of Neuhaus (1971), for every $\varepsilon>0$ and $\eta>0$, there exists a $\delta>0$, such that

(2.21) $$\lim_{n\to\infty} P\{\omega_\delta(W_n) > \varepsilon\} < \eta.$$

Hence, again using (2.18) and Rényi's (1958) idea of mixing sequence of sets, we have for $A\in\mathcal{A}$,

(2.22) $$\lim_{n\to\infty} P\{\omega_\delta(W_n) > \varepsilon|A\} < \eta.$$

3. The proof of Theorem 1. Let $[s]$ denote the largest integer $\leq s$. Then for $\varepsilon>0$,

$$P\{\rho(W_{N_\nu}, W_{[\nu\xi]}) > \varepsilon\} \tag{3.1}$$

$$\leq P\{|\nu^{-1}N_\nu-\xi|\geq\delta'\} + P\{\rho(W_{N_\nu}, W_{[\nu\xi]}) > \varepsilon,\ |\nu^{-1}N_\nu-\xi|<\delta'\}$$

$$\leq P\{|\nu^{-1}N_\nu-\xi|\geq\delta'\} + P\{\max_{k:\ |k-[\nu\xi]|<\delta'\nu} \rho(W_k, W_{[\nu\xi]})>\varepsilon\},\quad \delta'>0.$$

Thus, if $\xi=c$, a positive constant, with probability one [the case treated in Pyke (1968)], it readily follows from (1.6) and (2.19) that the right hand side of (3.1) can be bounded by $\eta(>0)$ by a proper choice of $\delta'>0$. The proof of the theorem then follows by noting that by the results of Neuhaus (1971), as $\nu\to\infty$,

$$W_{[\nu c]} \xrightarrow{\mathcal{D}} W, \text{ in the Skorokhod } J_1\text{-typology on } D^p[0,1]. \tag{3.2}$$

So, in the sequel, we consider the general case of ξ having an arbitrary distribution on $(0,\infty)$. For every $\eta>0$, there exists an $a_o = a_o(\eta)$, such that

$$P\{\xi \leq a_o(\eta)\} < \frac{1}{4}\eta. \tag{3.3}$$

Consider then a countable set of events

$$A_h = \{\xi:\ a_o(\eta)+h\delta'<\xi\leq a_o(\eta)+(h+1)\delta'\},\quad h=0,1,\ldots, \tag{3.4}$$

and let $a_h = a_h(\delta',\eta) = a_o(\eta)+(h+\frac{1}{2})\delta'$, $h=0,1,\ldots$ Then, the right hand side of (3.1) is bounded above by

$$(3.5)\qquad P\{|\nu^{-1}N_\nu-\xi|\ge\delta'\} + P\{\xi \le a_o(\eta)\} +$$

$$\sum_{h=0}^{\infty} P\{\max_{k:\ |k-[\nu\xi]|<\nu\delta'} \rho(W_k,W_{[\nu\xi]})>\varepsilon|A_h\}P(A_h)$$

$$\le P\{|\nu^{-1}N_\nu-\xi|\ge\delta'\} + P\{\xi \le a_o(\eta)\} +$$

$$\sum_{h=1}^{\infty} P\{\max_{k:\ |k-[\nu a_h]|<\frac{3}{2}\delta'\nu} \rho(W_k,W_{[\nu a_h]})>\varepsilon|A_h\}P(A_h).$$

Now, by (1.6) and (3.3), the first two terms on the right hand side of (3.5) are bounded by $\eta/4$ by proper choice of $\delta'>0$, while by (2.19), the last term can also be bounded by $\eta/2$, by proper choice of $\delta'(>0)$, as $\nu a_h\to\infty$ with $\nu\to\infty$, for every $h\ge 0$. Consequently, as $\nu\to\infty$,

$$(3.6)\qquad \rho(W_{N_\nu},W_{[\nu\xi]}) \xrightarrow{P} 0.$$

Thus, it suffices to show that as $\nu\to\infty$,

$$(3.7)\qquad W_{[\nu\xi]} \xrightarrow{\mathcal{D}} W,\ \text{in the Skorokhod } J_1\text{-topology on } D^p[0,1].$$

Now, (3.2) implies the convergence of the finite dimensional distributions of $\{W_\nu\}$ to those of W, while (2.13) implies that for any $\underline{t}\in E^p$, $\{W_\nu(\underline{t}): |\nu-n|<\delta n\}$ satisfy the "uniform continuity in probability" condition; these two conditions, in accordance with Theorem 1 of Mogyorodi (1965), imply the convergence of the finite dimensional distributions of $\{W_{[\nu\xi]}\}$ to those of W. So, to complete the proof of the theorem, we require to establish the 'tightness' property of $\{W_{[\nu\xi]}\}$ when $\nu\to\infty$. By (1.7) and (3.5) of Neuhaus (1971), it suffices to show that for every $\varepsilon>0$ and $\eta>0$, there exists a positive δ, such that as $\nu\to\infty$,

$$(3.8)\qquad P\{\omega_\delta(W_{[\nu\xi]}) \ge \varepsilon\} < \eta.$$

To show this, we note that for every $\varepsilon>0$, $\delta'>0$,

$$(3.9) \qquad P\{\omega_{\delta'}(W_{[\nu\xi]}) \geq \varepsilon\}$$

$$\leq P\{\xi \leq a_o(\eta)\} + P\{\omega_{\delta'}(W_{[\nu\xi]}) \geq \varepsilon,\ \xi > a_o(\eta)\}$$

$$= P\{\xi \leq a_o(\eta)\} + \sum_{h=0}^{\infty} P\{\omega_{\delta'}(W_{[\nu\xi]}) \geq \varepsilon | A_h\} P(A_h)$$

$$\leq P\{\xi \leq a_o(\eta)\} + \sum_{h=0}^{\infty} P\{\omega_{\delta'}(W_{[\nu a_h]}) \geq \tfrac{1}{3}\varepsilon | A_h\} P(A_h)$$

$$+ \sum_{h=0}^{\infty} P\{\rho(W_{[\nu\xi]}, W_{[\nu a_h]}) \geq \tfrac{1}{3}\varepsilon | A_h\} P(A_h)$$

$$\leq P\{\xi \leq a_o(\eta) + \sum_{h=0}^{\infty} P\{\omega_{\delta'}(W_{[\nu a_h]}) \geq \tfrac{1}{3}\varepsilon | A_h\} P(A_h)$$

$$+ \sum_{h=0}^{\infty} P\{\max_{k:\ |k-[\nu a_h]|<\frac{1}{2}\delta'\nu} \rho(W_k, W_{[\nu a_h]}) \geq \tfrac{1}{3}\varepsilon | A_h\} P(A_h),$$

which, by (3.3), (2.22) and (2.19), can be made smaller than $\eta(>0)$ by a proper choice of $\delta'(>0)$. Q.E.D.

REFERENCES

[1] BLUM. J. R., HANSON, D. L., AND ROSENBLATT, J. I. (1963). On the central limit theorem for the sum of a random number of independent random variables. Zeit. Wahrsch Verw. Geb. 1, 389-393.

[2] FELLER, W. (1966). An introduction to Probability Theory and its Applications, Vol 2. John Wiley, New York.

[3] KIEFER, J., AND WOLFOWITZ., J. (1958). On the deviations of the empiric distribution function of vector chance variables. Trans. Amer. Math. Soc. 87, 173-186.

[4] MOGYORODI, J. (1965). Limit distributions for sequences of random variables with random indices. Trans. 4th Prague Confer. Infor. Th. Statist. Dec. fn. Random. Proc. 463-470.

[5] NEUHAUS, G. (1971). On weak convergence of stochastic processes with multidimensional time parameter. Ann. Math. Statist. 42, 1285-1295.

[6] PYKE, R. (1968). The weak convergence of the empirical process with random sample size. Proc. Cambridge Phil. Soc. 64, 155-160.

[7] RÉNYI, A. (1958). On mixing sequences of sets. Acta Math. Acad. Sc. Hungar. 9, 215-228.